Spatial correlations in sheared isothermal liquids: From elastic particles to granular particles

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Spatial correlations in sheared isothermal liquids for both elastic and granular cases are theoretically investigated. Using the generalized fluctuating hydrodynamics, correlation functions for both the microscopic scale and the macroscopic scale are obtained. We find the existence of long-range correlations obeying power laws. The validity of our theoretical predictions has been verified from molecular-dynamics simulation.

I. INTRODUCTION

Liquids consist of assemblies of many particles. There is long history of study of molecular liquids, which consist of nondissipative simple molecules [1,2]. Recently, there has been a rapidly growing interest in granular liquids, which are made of granular assemblies, i.e., dissipative macroscopic particles [3–18]. Although a molecular liquid is fluctuated around an equilibrium state, the absence of the equilibrium state of granular assemblies makes granular materials unlike usual materials [19]. Nevertheless, we have recognized that statistical properties and hydrodynamic behaviors of moderate dense and nearly elastic granular gases are considerably well understood from the analysis of the kinetic theory [9,13,20–24].

Boltzmann-Enskog theory is often used in describing moderate dense gases including granular assemblies, where the assumption of molecular chaos is used [9,20–24]. In spite of the success of qualitative description of hydrodynamic behaviors based on Boltzmann-Enskog theory, significant roles of correlations have lately been recognized. Indeed, it is well known that correlated collisions cause significant differences in constitutive equations even in equilibrium gases. In particular, recently a number of papers on long-time tails in current correlation functions for granular liquids have been published, which are directly related to the transport coefficients [11,12,15,16,18,25–27]. On the other hand, we know the existence of long-range correlations in sheared elastic fluids [28–30] and heat conduction systems [31–37]. The existence of a similar long-range correlation has been observed even in a simulation of sheared granular fluids [38]. In spite of the indication of the existence of the long-range correlations in sheared fluids, we do not have any consistent theory in describing a structure factor for both the particle scale and the hydrodynamic scale. Indeed, a theoretical prediction of spatial correlations in the molecular scale for relatively dense liquids under a shear is not consistent with the long-range correlations [39]. Thus, we still do not understand the details of spatial correlations even in the case of molecular liquids.

We often use the mode-coupling theory (MCT) in describing dense liquids [40–42]. Even when we are interested in sheared dense granular liquids, the framework of MCT can be used [17]. However, MCT is not a closed theory, because it needs to determine the structure factor by another method. Thus, to obtain the structure factor or the pair-correlation function is an important issue for the description of dense granular liquids.

The purpose of this paper is, thus, to clarify the spatial correlation functions for sheared isothermal liquids. Based on the generalized fluctuating hydrodynamics [43,44], we will demonstrate that the long-range correlation obeying a power law is consistent with the short-range structure obtained by a liquid theory. In the next section, we will summarize the outline of the generalized fluctuating hydrodynamics. In Sec. III, we will show properties of a set of linearized equations around a uniform shear flow (USF) based on the generalized fluctuating hydrodynamics. In Sec. IV, we will calculate the spatial correlation functions and their asymptotic forms in the long-range limit. In Sec. V, we will compare our results with those of the molecular-dynamics simulation. In Sec. VI, we will discuss and conclude our results. In Appendix A, we summarize the form of the pair-correlation function $g_0(r, e)$ for homogeneous unsheared states obtained by Lutsko [6]. In Appendix B, we briefly summarize the transformation between the Cartesian coordinate and the oblique coordinate. In Appendix C, we present the explicit form of $\widetilde{C}_{nn}(\boldsymbol{k})$, which is the Fourier component of the density correlation. In Appendix D, we show the explicit form of $\widetilde{C}_{pp}(\boldsymbol{k})$, which is the Fourier component of the momentum correlation. In Appendix E, we evaluate the asymptotic form of a function $\overline{\Delta}_j(r)$, which is needed for the calculation of correlation functions.

II. GENERALIZED FLUCTUATING HYDRODYNAMICS

We consider three-dimensional systems consisting of N identical smooth and hard spherical particles confined in the volume $V = L^3$ under a shear flow with the shear rate $\dot{\gamma}$, where each particle has mass m and diameter σ. The particles collide instantaneously with each other by a restitution constant e that is less than unity for granular particles and equal to unity for molecular liquids. Let us assume that the restitution coefficient e is a constant.

The spatial correlations in sheared fluids are investigated by using the fluctuating hydrodynamics [28–30,38], which

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can be used for the description of hydrodynamic behaviors. However, it is possible to generalize the fluctuating hydrodynamics that can cover the scale around the particle diameter. The generalized hydrodynamic equations [43–46] for isothermal liquids characterized by a uniform temperature T are given by

$$\partial_t n + \nabla \cdot (n\boldsymbol{u}) = 0, \tag{1}$$

$$\partial_t u_\alpha + u_\alpha \nabla_\beta u_\beta + \frac{1}{m}\nabla_\alpha \mu + \frac{1}{mn}\nabla_\beta (\Sigma_{\alpha\beta}^D + \Sigma_{\alpha\beta}^R) = 0, \tag{2}$$

where n and u_α are the number density and α-component of the velocity field, respectively. Here, we have introduced the generalized chemical potential or the effective pressure

$$\mu = T\left[\ln n - \int d\boldsymbol{r}' C(\boldsymbol{r} - \boldsymbol{r}', e, \dot{\gamma}) \delta n(\boldsymbol{r}', t) + \cdots \right], \tag{3}$$

where $C(\boldsymbol{r} - \boldsymbol{r}', e, \dot{\gamma})$ is the two-particle direct correlation function, which satisfies $nC(k, e, \dot{\gamma}) \equiv 1 - S(k, e, \dot{\gamma})^{-1}$ with the structure factor $S(k, e, \dot{\gamma})$. $\Sigma_{\alpha\beta}^D(\boldsymbol{r}, t)$ is the viscous stress tensor given by

$$\Sigma_{\alpha\beta}^D(\boldsymbol{r}, t) = -\int d\boldsymbol{r}'\left[\eta(\boldsymbol{r} - \boldsymbol{r}', e)\left\{ \dot{\epsilon}_{\alpha\beta}(\boldsymbol{r}', t) + \dot{\epsilon}_{\beta\alpha}(\boldsymbol{r}', t) \right. \right.$$
$$\left. \left. - \frac{2}{3}\dot{\epsilon}_{\gamma\gamma}(\boldsymbol{r}', t)\delta_{\alpha\beta} \right\} - \zeta(\boldsymbol{r} - \boldsymbol{r}', e)\dot{\epsilon}_{\gamma\gamma}(\boldsymbol{r}', t)\delta_{\alpha\beta} \right], \tag{4}$$

where $\dot{\epsilon}_{\alpha\beta}(\boldsymbol{r}, t) \equiv \{\nabla_\alpha u_\beta(\boldsymbol{r}, t) + \nabla_\beta u_\alpha(\boldsymbol{r}, t)\}/2$. $\Sigma_{\alpha\beta}^R(\boldsymbol{r}, t)$ is the random part of the stress tensor satisfying $\langle \Sigma_{\alpha\beta}^R(\boldsymbol{r}, t) \rangle = 0$, and

$$\langle \Sigma_{\alpha\beta}^R(\boldsymbol{r}, t)\Sigma_{\gamma\delta}^R(\boldsymbol{r}', t') \rangle = 2T\delta(t - t')\{\eta(\boldsymbol{r} - \boldsymbol{r}', e)\Delta_{\alpha\beta\gamma\delta}$$
$$+ \zeta(\boldsymbol{r} - \boldsymbol{r}', e)\delta_{\alpha\beta}\delta_{\gamma\delta}\} \tag{5}$$

with $\Delta_{\alpha\beta\delta\gamma} \equiv \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - 2\delta_{\alpha\beta}\delta_{\gamma\delta}/3$. Here, we have used Einstein's sum rule on the Greek subscript. The generalized shear viscosity $\eta(\boldsymbol{r}, e)$ and the generalized bulk viscosity $\zeta(\boldsymbol{r}, e)$ are represented by $\nu_1^*(k, e)$, $\nu_2^*(k, e)$ and Enskog's mean free time

$$t_E \equiv \frac{1}{4\pi n_0 \sigma^2 g_0(\sigma, e)}\left(\frac{m\pi}{T} \right)^{1/2} \tag{6}$$

with the radial distribution function $g_0(\sigma, e)$ at contact as $\nu_1^*(k, e) = (mn_0\sigma^2 t_E^{-1})^{-1}[\zeta(k, e) + 4\eta(k, e)/3]$, and $\nu_2^*(k, e) = (mn_0\sigma^2 t_E^{-1})^{-1}\eta(k, e)$, where n_0, $\eta(k, e)$, and $\zeta(k, e)$ are, respectively, the average number density, and Fourier transforms of $\eta(\boldsymbol{r}, e)$ and $\zeta(\boldsymbol{r}, e)$. It is known that $\nu_1^*(k, e)$ and $\nu_2^*(k, e)$ are given, respectively, by

$$\nu_1^*(k, 1) = 2[1 - j_0(k) + 2j_2(k)]/(3k^2) \tag{7}$$

and

$$\nu_2^*(k, 1) = 2[1 - j_0(k) - j_2(k)]/(3k^2) \tag{8}$$

for elastic hard spherical particles, where $j_l(k)$ with $l = 0$ or 2 is the lth-order spherical Bessel function [43–48]. Although we do not know how $\nu_1^*(k, e)$ and $\nu_2^*(k, e)$ depend on e, the

explicit e-dependences of $\nu_1^*(k, e)$ and $\nu_2^*(k, e)$ are not important in this paper. Therefore, we are omitting their explicit forms from the discussion.

It should be noted that sheared corrections to the structure factor or the pair-correlation function can be obtained within this theoretical framework, though the equilibrium or the unsheared structure factor $S_0(k, e) \equiv S(k, e, \dot{\gamma} = 0)$ should be determined by another method. We adopt an approximate expression of the pair-correlation function for unsheared granular liquids obtained by Lutsko [6], which covers the equilibrium pair correlation in the elastic limit (see Appendix A).

This set of Eqs. (1)–(5) is a reasonable starting point for isothermal molecular liquids ($e = 1$), once we use appropriate generalized transport coefficients and $S_0(k, e = 1)$. However, the validity of this set of equations in describing isothermal sheared granular fluids might be controversial. Indeed, nobody has used the generalized fluctuating hydrodynamics for granular liquids, because we know that fluctuations cannot be characterized by a Gaussian. In addition, there are no explicit characteristics of granular liquids in the set of equations except for e-dependence of $S(k, e, \dot{\gamma})$ and the generalized transport coefficients. Even when we use the hydrodynamic equations for granular liquids, we should consider an equation for the granular temperature.

Let us answer the above critical points to validate the generalized fluctuating hydrodynamics in Eqs. (1)–(5). First, the granular temperature T is not a true hydrodynamic variable but a relatively fast variable because of the collisional energy loss in granular systems. Thus, we expect the fast relaxation to a steady state of the temperature for sheared granular liquids, which satisfies $T \propto \dot{\gamma}^2/(1 - e^2)$. Second, an isothermal situation is easily realized by the balance between the viscous heating and the collisional energy loss. In particular, it is known that USF is stable for small and nearly elastic systems under the Lees-Edwards boundary condition. Even in physical situations, the heat conduction is not important in the bulk region far from the boundary. In these situations, we may assume that sheared granular liquids are nearly isothermal. We also indicate that our previous studies clarify the formal similarities between sheared granular liquids and sheared molecular liquids at a constant temperature [16,17]. Through previous studies, we have recognized that the most important issue is to determine $S(k, e, \dot{\gamma})$ for granular liquids [17], which can be determined within the framework of the generalized fluctuating hydrodynamics. Third, the fluctuating hydrodynamics has been used in describing granular hydrodynamics for freely cooling cases [49]. Thus, we believe that the set of Eqs. (1)–(5) can be used even for sheared granular liquids. Although there are uncovered regions of our approach in the description of granular liquids, we expect that our approach can capture some aspects of sheared granular liquids. The validity of the model will be tested from the comparison between our theoretical prediction and the direct simulation of granular assemblies.

III. LINEARIZED EQUATIONS AROUND UNIFORM SHEAR FLOW AND THEIR SOLUTIONS

In this section, we analyze a set of the linearized equations around USF. As mentioned in the previous section, we

assume that USF is stable. Thus, we only need to solve the linearized equations. This section consists of two parts. In the first part, we summarize the expression of the linearized equations. In the second part, we explicitly write the solutions of the linearized equations.

A. Linearized equations

Let us introduce the fluctuations of the hydrodynamic fields $\delta n(\mathbf{r},t) \equiv n(\mathbf{r},t) - n_0$, $\delta \mathbf{u}(\mathbf{r},t) \equiv \mathbf{u}(\mathbf{r},t) - \mathbf{c}(\mathbf{r},t)$ with $c_\alpha(\mathbf{r}) = \dot{\gamma} y \delta_{\alpha,x}$, and the nondimensionalized vector $\mathbf{z}(\mathbf{r},t)$, whose Fourier transform is given by

$$\hat{z}^T(\mathbf{q},t) = (\delta n(\mathbf{q},t), \delta u_x(\mathbf{q},t)/(t_E^{-1}\sigma^4),$$
$$\delta u_y(\mathbf{q},t)/(t_E^{-1}\sigma^4), \delta u_z(\mathbf{q},t)/(t_E^{-1}\sigma^4)). \quad (9)$$

Here, the Greek suffix α denotes the Cartesian component.

$$L_0 = \begin{bmatrix} 0 & n_0\sigma^3 i k_x & n_0\sigma^3 i k_y & n_0\sigma^3 i k_z \\ p^* i k_x & (\nu_1^* - \nu_2^*)k_x^2 + \nu_2^* k^2 & (\nu_1^* - \nu_2^*)k_x k_y & (\nu_1^* - \nu_2^*)k_x k_z \\ p^* i k_y & (\nu_1^* - \nu_2^*)k_y k_x & (\nu_1^* - \nu_2^*)k_y^2 + \nu_2^* k^2 & (\nu_1^* - \nu_2^*)k_y k_z \\ p^* i k_z & (\nu_1^* - \nu_2^*)k_z k_x & (\nu_1^* - \nu_2^*)k_z k_y & (\nu_1^* - \nu_2^*)k_z^2 + \nu_2^* k^2 \end{bmatrix}, \quad (12)$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (13)$$

with $p_1^* \equiv p^*(k,e) = AS_0(k,e)^{-1}$ and $A \equiv T/(mn_0\sigma^5 t_E^{-2})$. The random vector $\widetilde{\mathbf{R}}$ has four components

$$\widetilde{R}_1 = 0,$$

$$\widetilde{R}_{\alpha+1} = (mn_0\sigma^4 t_E^{-2})^{-1} i \sigma k_\beta \Sigma_{\alpha\beta}^R(\mathbf{k},\bar{t}), \quad (14)$$

where $\alpha = 1, 2, 3$, respectively, correspond to x, y, and z. Although the $\dot{\gamma}$ dependence of $S(k,e,\dot{\gamma})$ should appear in L_1, we simply ignore such terms. The validity of this simplification will be checked from a comparison of the results with our simulation.

B. Solution of the linearized equations

It is straightforward to solve Eq. (10). As mentioned in Ref. [28], its calculation can be simplified if we introduce the transformation from the Cartesian coordinate to the oblique coordinate to decompose the longitudinal modes and the transverse mode. However, the obtained results in the oblique coordinate are rather confusing, because of too many suffixes. In addition, the results are basically the same as those obtained by Lutsko and Dufty [28] with replacing the transport coefficients by the generalized transport coefficients, and ignoring terms related with the fluctuation of the

From Eqs. (1), (2), and (9), we obtain the linearized evolution equation for $\widetilde{z}(\mathbf{k},\bar{t}) \equiv \hat{z}(\mathbf{q},t)$ as

$$\left(\partial_{\bar{t}} - \dot{\gamma}^* k_x \frac{\partial}{\partial k_y}\right)\widetilde{z} + L \cdot \widetilde{z} = \widetilde{\mathbf{R}}, \quad (10)$$

where the time, the wave number, and the shear rate have been nondimensionalized by $t = t_E \bar{t}$, $\mathbf{q} = \mathbf{k}/\sigma$, and $\dot{\gamma} = \dot{\gamma}^*/t_E$, respectively. Here, the matrix L is expanded as

$$L = L_0 + \dot{\gamma}^* L_1 + \cdots, \quad (11)$$

where L_0 and L_1 are given, respectively, by

temperature. In this paper, therefore, we only present the final results in the Cartesian coordinate. The transformation between the Cartesian coordinate and the oblique coordinate is explained in Appendix B.

The solution of Eq. (10) can be formally represented by

$$\widetilde{z}(\mathbf{k},\bar{t}) = \sum_{j=1}^{4} \int_{-\infty}^{\bar{t}} ds \, \widetilde{\psi}^{(j)}(\mathbf{k},\bar{t}-s) F^{(j)}(\widetilde{\mathbf{k}}[\dot{\gamma}^*(s-\bar{t})],s), \quad (15)$$

where

$$\widetilde{\psi}^{(j)}(\mathbf{k},\bar{t}) \equiv \psi^{(j)}(\mathbf{k}) \exp\left(-\int_0^{\bar{t}} d\bar{\tau} \lambda^{(j)}[\widetilde{\mathbf{k}}(\dot{\gamma}^*\bar{\tau})]\right), \quad (16)$$

$$F^{(j)}(\mathbf{k},\bar{t}) \equiv \varphi^{(j)}(\mathbf{k}) \cdot \widetilde{\mathbf{R}}(\mathbf{k},\bar{t}) \quad (17)$$

with $\widetilde{\mathbf{k}}(\bar{\tau}) \equiv (k_x, k_y + \bar{\tau} k_x, k_z)$. Here, we have introduced the right eigenvectors $\psi^{(j)}(\mathbf{k})$, the associated biorthogonal vectors, i.e., the left eigenvectors $\varphi^{(j)}(\mathbf{k})$, and the eigenvalues $\lambda^{(j)}(\mathbf{k})$ satisfying

$$\left(-1\dot{\gamma}^* k_x \frac{\partial}{\partial k_y} + L\right) \cdot \psi^{(j)}(\mathbf{k}) = \lambda^{(j)}(\mathbf{k})\psi^{(j)}(\mathbf{k}). \quad (18)$$

We also note that $\psi^{(j)}(\mathbf{k})$ and $\varphi^{(j)}(\mathbf{k})$ satisfy $\psi^{(i)}(\mathbf{k}) \cdot \varphi^{(j)}(\mathbf{k}) = \delta_{ij}$.

In order to obtain $\psi^{(j)}(\mathbf{k})$, $\varphi^{(j)}(\mathbf{k})$, and $\lambda^{(j)}(\mathbf{k})$, we use the expansions

$$\psi^{(j)} = \psi_0^{(j)} + \dot{\gamma}^* \psi_1^{(j)} + \cdots, \quad (19)$$

$$\boldsymbol{\varphi}^{(j)} = \boldsymbol{\varphi}_0^{(j)} + \dot{\gamma}^* \boldsymbol{\varphi}_1^{(j)} + \cdots , \tag{20}$$

$$\lambda^{(j)} = \lambda_0^{(j)} + \dot{\gamma}^* \lambda_1^{(j)} + \cdots , \tag{21}$$

in terms of $\dot{\gamma}^*$.

We should note that the perturbation in terms of $\dot{\gamma}^*$ is not the expansion from an unsheared state of granular liquids. Indeed, it is well known that properties of sheared granular liquids differ completely from those of freely cooling granular liquids. In the case of sheared granular liquids, we obtain the relation $\dot{\gamma}^* \sim \dot{\gamma}/\sqrt{T} \sim \sqrt{1-e^2}$ from the balance between the viscous heating and the collisional energy loss. Thus, the expansion in terms of $\dot{\gamma}^*$ can be regarded as that by small inelasticity [50].

Substituting Eqs. (19)–(21) into Eq. (18), we obtain the zeroth- and the first-order perturbations as

$$(\mathsf{L}_0 - \lambda_0^{(j)}1) \cdot \boldsymbol{\psi}_0^{(j)}(\boldsymbol{k}) = 0, \tag{22}$$

$$(\mathsf{L}_{(0)} - \lambda_0^{(j)}1) \cdot \boldsymbol{\psi}_1^{(j)}(\boldsymbol{k}) + \left(-1 k_x \frac{\partial}{\partial k_y} + \mathsf{L}_1 - \lambda_1^{(j)}1 \right) \cdot \boldsymbol{\psi}_0^{(j)}(\boldsymbol{k}) = 0. \tag{23}$$

Solving these equations, we obtain the eigenvalues

$$\lambda^{(1)} = \lambda_+ + \dot{\gamma}^* \frac{k_x k_y}{k^2} \xi^{(1)}(k), \tag{24}$$

$$\lambda^{(2)} = \lambda_- + \dot{\gamma}^* \frac{k_x k_y}{k^2} \xi^{(2)}(k), \tag{25}$$

$$\lambda^{(3)} = \nu_2^*(k,e)k^2 - \dot{\gamma}^* \frac{k_x k_y}{k^2}, \tag{26}$$

$$\lambda^{(4)} = \nu_2^*(k,e)k^2 \tag{27}$$

within the approximation up to $O(\dot{\gamma}^*)$, where we have introduced

$$\lambda_+ = \frac{\nu_1^*(k,e)k^2 + \sqrt{[\nu_1^*(k,e)k^2]^2 - 4n_0\sigma^3 p^*(k,e)k^2}}{2}, \tag{28}$$

$$\lambda_- = \frac{\nu_1^*(k,e)k^2 - \sqrt{[\nu_1^*(k,e)k^2]^2 - 4n_0\sigma^3 p^*(k,e)k^2}}{2}, \tag{29}$$

$$\xi^{(1)}(k) \equiv \frac{\lambda_+^2}{N_+^2} + \frac{n_0\sigma^3 k^2}{2N_+^2} k\partial_k p^*(k,e),$$

$$\xi^{(2)}(k) \equiv \frac{\lambda_-^2}{N_-^2} + \frac{n_0\sigma^3 k^2}{2N_-^2} k\partial_k p^*(k,e), \tag{30}$$

and

$$N_+^2 = -n_0\sigma^3 p^*(k,e)k^2 + \lambda_+^2, \quad N_-^2 = -n_0\sigma^3 p^*(k,e)k^2 + \lambda_-^2. \tag{31}$$

Similarly, we obtain the right eigenvectors

$$\boldsymbol{\psi}^{(1)T} = \frac{1}{N_+}\left(ikn_0\sigma^3, \lambda_+\frac{k_x}{k}, \lambda_+\frac{k_y}{k}, \lambda_+\frac{k_z}{k} \right), \tag{32}$$

$$\boldsymbol{\psi}^{(2)T} = \frac{1}{N_-}\left(ikn_0\sigma^3, \lambda_-\frac{k_x}{k}, \lambda_-\frac{k_y}{k}, \lambda_-\frac{k_z}{k} \right), \tag{33}$$

$$\boldsymbol{\psi}^{(3)} = \boldsymbol{\Psi}^{(3)} + M(\boldsymbol{k})\boldsymbol{\Psi}^{(4)}, \tag{34}$$

$$\boldsymbol{\psi}^{(4)} = \boldsymbol{\Psi}^{(4)}, \tag{35}$$

and the left eigenvectors

$$\boldsymbol{\varphi}^{(1)} = \frac{1}{N_+}\left(ikp^*(k,e), \lambda_+\frac{k_x}{k}, \lambda_+\frac{k_y}{k}, \lambda_+\frac{k_z}{k} \right), \tag{36}$$

$$\boldsymbol{\varphi}^{(2)} = \frac{1}{N_-}\left(ikp^*(k,e), \lambda_-\frac{k_x}{k}, \lambda_-\frac{k_y}{k}, \lambda_-\frac{k_z}{k} \right), \tag{37}$$

$$\boldsymbol{\varphi}^{(3)} = \boldsymbol{\Phi}^{(3)}, \tag{38}$$

$$\boldsymbol{\varphi}^{(4)} = -M(\boldsymbol{k})\boldsymbol{\Phi}^{(3)} + \boldsymbol{\Phi}^{(4)}, \tag{39}$$

where we have used

$$\boldsymbol{\Psi}^{(3)T} = \boldsymbol{\Phi}^{(3)} \equiv \left(0, -\frac{k_y k_x}{kk_\perp}, \frac{k_\perp}{k}, -\frac{k_y k_x}{kk_\perp} \right), \tag{40}$$

$$\boldsymbol{\Psi}^{(4)T} = \boldsymbol{\Phi}^{(4)} \equiv \left(0, \frac{k_z}{k_\perp}, 0, -\frac{k_x}{k_\perp} \right), \tag{41}$$

and

$$M(\boldsymbol{k}) = -\frac{kk_z}{k_x k_\perp}\tan^{-1}(k_y/k_\perp) \tag{42}$$

with $k_\perp \equiv k^2 - k_y^2$.

IV. CORRELATION FUNCTIONS

This section is the main part of this paper, in which we present the explicit forms of correlation functions. This section consists of two parts. The first part summarizes the general results of correlation functions. In the second part, we evaluate integrals included in correlation functions to extract the long-range behaviors of correlations.

A. General results for correlation functions

Let us introduce the correlation functions $\widetilde{C}_{ii}(\boldsymbol{k},\bar{t})$, which satisfy

$$\langle \widetilde{z}_i(\boldsymbol{k},\bar{t})\widetilde{z}_i(\boldsymbol{k}',\bar{t}) \rangle = (2\pi)^3 \delta^3(\boldsymbol{k}+\boldsymbol{k}')\widetilde{C}_{ii}(\boldsymbol{k},\bar{t}). \tag{43}$$

Note that we do not use Einstein's sum rule for Latin subscripts. Substituting Eq. (15) into this equation, we obtain

$$\widetilde{C}_{ii}(\boldsymbol{k},\bar{t}) = \int_0^\infty d\bar{t} \sum_{l,m} \widetilde{\psi}_i^{(l)}(\boldsymbol{k},\bar{t})\widetilde{\psi}_i^{(m)}(-\boldsymbol{k},\bar{t})F^{(lm)}[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{t})], \tag{44}$$

where $\widetilde{\psi}_i^{(l)}(\boldsymbol{k},\bar{t})$ is the ith. component of $\widetilde{\boldsymbol{\psi}}^{(l)}(\boldsymbol{k},\bar{t})$, and $F^{(lm)}(\boldsymbol{k})$ satisfies

$$\langle F^{(l)}(\boldsymbol{k},\bar{t})F^{(m)}(\boldsymbol{k}',\bar{t}')\rangle = (2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}')\,\delta(\bar{t}-\bar{t}')F^{(lm)}(\boldsymbol{k}).$$

$$(45)$$

From Eqs. (14), (17), and (36)–(39), the explicit forms of $F^{(lm)}(\boldsymbol{k})$ are given by

$$F^{(11)}(\boldsymbol{k}) = -2Ak^2\nu_1^*(k,e)\frac{\lambda_+^2}{N_+^2},$$

$$F^{(22)}(\boldsymbol{k}) = -2Ak^2\nu_1^*(k,e)\frac{\lambda_-^2}{N_-^2},$$

$$F^{(12)}(\boldsymbol{k}) = F^{(21)}(\boldsymbol{k},\bar{t}) = -2Ak^2\nu_1^*(k,e)\frac{\lambda_+\lambda_-}{N_+N_-},$$

$$F^{(33)}(\boldsymbol{k}) = 2Ak^2\nu_2^*(k,e),$$

$$F^{(44)}(\boldsymbol{k}) = -[M(\boldsymbol{k})^2 + 1]F^{(33)}(\boldsymbol{k}),$$

$$F^{(34)}(\boldsymbol{k}) = -F^{(43)}(\boldsymbol{k}) = M(\boldsymbol{k})F^{(33)}(\boldsymbol{k}).$$

$$(46)$$

Thus, we can calculate any spatial correlation functions.

Let us explicitly write the spatial density correlation

$$C_{nn}(\boldsymbol{r},\bar{t}) \equiv \langle \delta n(\boldsymbol{r}+\boldsymbol{r}',\bar{t})\delta n(\boldsymbol{r}',\bar{t})\rangle = \sigma^{-6}C_{11}(\boldsymbol{r},\bar{t})$$

$$(47)$$

and the spatial momentum correlation

$$C_{pp}(\boldsymbol{r},\bar{t}) \equiv \langle \boldsymbol{p}(\boldsymbol{r}+\boldsymbol{r}',\bar{t})\cdot\boldsymbol{p}(\boldsymbol{r}',\bar{t})\rangle$$

$$\simeq (mn_0\sigma/t_E)^2\{\tilde{C}_{22}(\boldsymbol{k},\bar{t})+\tilde{C}_{33}(\boldsymbol{k},\bar{t})\tilde{C}_{44}(\boldsymbol{k},\bar{t})\}$$

$$(48)$$

with the momentum density $\boldsymbol{p}(\boldsymbol{r},\bar{t})=mn(\boldsymbol{r},\bar{t})\delta\boldsymbol{u}(\boldsymbol{r},\bar{t})$. In Eq. (48), we adopt the approximation $\boldsymbol{p}(\boldsymbol{r},\bar{t})\simeq mn_H\delta\boldsymbol{u}(\boldsymbol{r},\bar{t})$. With the help of the inverse Fourier transform

$$C_{nn}(\boldsymbol{r},\bar{t}) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\tilde{C}_{nn}(\boldsymbol{k},\bar{t})e^{-i\boldsymbol{q}\cdot\boldsymbol{r}},$$

$$(49)$$

$$C_{pp}(\boldsymbol{r},\bar{t}) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\tilde{C}_{pp}(\boldsymbol{k},\bar{t})e^{-i\boldsymbol{q}\cdot\boldsymbol{r}},$$

$$(50)$$

and Eq. (44), we obtain the steady solutions (see Appendixes C and D)

$$\tilde{C}_{nn}(\boldsymbol{k}) = n_0\{S_0(k,e)+\dot{\gamma}^*\tilde{\Delta}_1(\boldsymbol{k})\},$$

$$(51)$$

$$\tilde{C}_{pp}(\boldsymbol{k}) = (mn_0\sigma/t_E)^2\{-\tilde{D}_2(\boldsymbol{k})+\tilde{D}_3(\boldsymbol{k})-\tilde{D}_4(\boldsymbol{k})\},$$

$$(52)$$

where $\tilde{C}_{nn}(\boldsymbol{k}) \equiv \lim_{\bar{t}\to\infty}\tilde{C}_{nn}(\boldsymbol{k},\bar{t})$, and $S_0(k,e)$ is the structure factor for the unsheared case. $\tilde{D}_j(\boldsymbol{k})$ with $j=2,3,4$ in Eq. (52) are given by

$$\tilde{D}_2(\boldsymbol{k}) = -A\{1+\dot{\gamma}^*\tilde{\Delta}_2(\boldsymbol{k})\},$$

$$(53)$$

$$\tilde{D}_3(\boldsymbol{k}) = A\{1+2\dot{\gamma}^*\tilde{\Delta}_3(\boldsymbol{k})\},$$

$$(54)$$

$$\tilde{D}_4(\boldsymbol{k}) = -A\{1+2\dot{\gamma}^*\tilde{\Delta}_4(\boldsymbol{k})\}.$$

$$(55)$$

The derivation of Eqs. (51)–(55) and the explicit forms of $\tilde{\Delta}_j(\boldsymbol{k},\bar{t})$ with $j=1,2,3,4$ are presented in Appendixes C and D.

From Eqs. (49)–(55), we finally obtain

$$C_{nn}(\boldsymbol{r}) = n_0^2\{g_0(r,e)-1+\dot{\gamma}^*\Delta_1(\boldsymbol{r})\}+n_0\delta(\boldsymbol{r}),$$

$$(56)$$

$$C_{pp}(\boldsymbol{r}) = \frac{T_0}{mn_0}[3\delta(\boldsymbol{r})+\sigma^{-3}\dot{\gamma}^*\{\Delta_2(\boldsymbol{r})+\Delta_3(\boldsymbol{r})+\Delta_4(\boldsymbol{r})\}],$$

$$(57)$$

where $g_0(r,e)$ is the pair-correlation function for unsheared cases, and

$$\Delta_\alpha(\boldsymbol{r}) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\tilde{\Delta}_\alpha(\boldsymbol{k})e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}.$$

$$(58)$$

The determination of $g_0(r,e)$ or $S_0(k,e)$ for $e<1$ is highly nontrivial. Indeed, we cannot keep any homogeneous cooling state (HCS) without artificial controls of the systems. However, Lutsko [6] obtained an approximate expression of $g_0(r,e)$ for HCS. He also verified that his approximate expression works well from the comparison between the theory and the simulation of HCS. As stated in Sec. II, we adopt his expression in this paper (Appendix A). We also note that the structure of the liquids in Eq. (56) can be represented by the linear contribution of the homogeneous terms and the sheared term.

B. Long-range correlation

Let us demonstrate the existence of the long-range correlation in $C_{nn}(\boldsymbol{r})$ and $C_{pp}(\boldsymbol{r})$. Let the angular average of any function $f(\boldsymbol{r})$ be denoted by $\bar{f}(r) \equiv \int d\Omega f(\boldsymbol{r})/(4\pi)$. As shown in Appendix E, the asymptotic forms of $\bar{\Delta}_j(r)$ ($j=1,2,3$) satisfy

$$\bar{\Delta}_1(r) \propto r^{-11/3}, \quad r \gg l_c, \quad (59)$$

$$\bar{\Delta}_2(r) \propto r^{-11/3}, \quad r \gg l_c, \quad (60)$$

$$\bar{\Delta}_3(r) \propto r^{-5/3}, \quad r \gg l_c, \quad (61)$$

$$\bar{\Delta}_4(r) \propto r^{-5/3}, \quad r \gg l_c, \quad (62)$$

where $l_c \equiv \sigma/\sqrt{\dot{\gamma}^*}$. Substituting these results into Eqs. (56) and (57), the long-range parts of $\bar{C}_{nn}(r)$ and $\bar{C}_{pp}(r)$, respectively, satisfy

$$\bar{C}_{nn}(r) \propto \left(\frac{r}{l_c}\right)^{-11/3}, \quad r \gg l_c, \quad (63)$$

$$\bar{C}_{pp}(r) \propto \left(\frac{r}{l_c}\right)^{-5/3}, \quad r \gg l_c. \quad (64)$$

This long-range correlation $C_{pp}(\boldsymbol{r})$ is known for isothermal sheared elastic fluids [28–30], and has been verified in sheared dilute granular fluids [38].
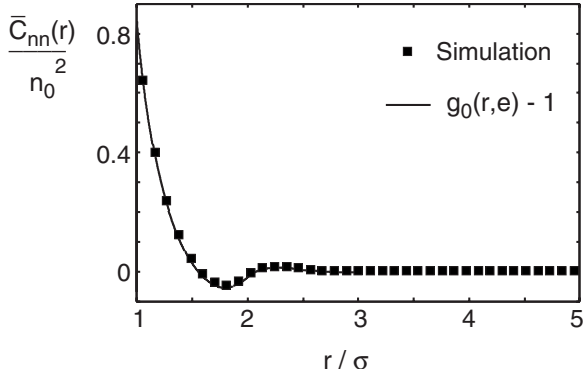
FIG. 1. The density correlation function $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.185$ with $L=89\sigma$ and $e=0.83$ as a function of the distance r. The solid line represents $g_0(r,e)$ obtained by Lutsko [6].

V. COMPARISON BETWEEN THEORY AND SIMULATION

To verify the validity of our theoretical prediction, we perform the event-driven molecular-dynamics simulation for three-dimensional hard spheres. In our simulation, the time scale is measured by $\tau_0 \equiv \sigma\sqrt{m/T_0}$, where T_0 is the averaged initial temperature. Particles are confined in a cell under the Lees-Edwards boundary condition, in which each linear dimension is L. The number of particles is not fixed in our simulation, but we control the system size L and the volume fraction ϕ as well as the restitution constant e. The initial state at the time $t=0$ is the equilibrium state, and we will show the correlation functions at $t=20\tau_0$ for $\phi=0.50$ and 0.37, and $t=40\tau_0$ for $\phi=0.185$ as $\bar{C}_{nn}(r)$ and $\bar{C}_{pp}(r)$, where the system is considered in a steady state. We also choose the shear rate to keep the steady temperature unity in the dimensionless unit, except for the data of Fig. 4, where the temperature in the steady state is $1.0T_0$ for $\dot{\gamma}=0.92\tau_0^{-1}$ or $4.0T_0$ for $\dot{\gamma}=1.84\tau_0^{-1}$.

Figures 1–3 show the behaviors of $\bar{C}_{nn}(r)$ for $r \lesssim 5\sigma$ at $\phi=0.185$, 0.37, and 0.50, respectively. The restitution coefficient e is 0.83 for Fig. 1, and e is 0.90 for Figs. 2 and 3. The solid line represents $g_0(r,e)$ obtained by Lutsko [6] without any fitting parameters. Although we omit contributions of
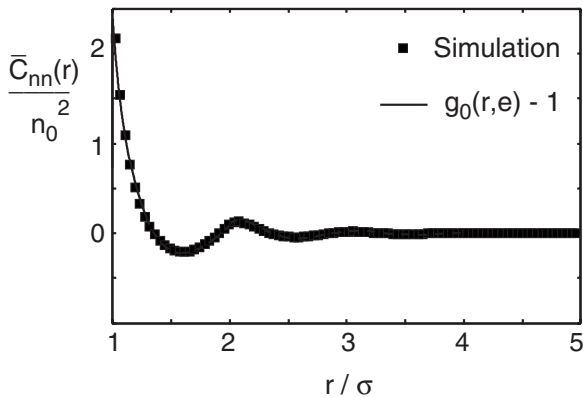

FIG. 2. The density correlation function $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.37$ with $L=72\sigma$ and $e=0.90$ as a function of the distance r. The solid line expresses $g_0(r,e)$ obtained by Lutsko [6].
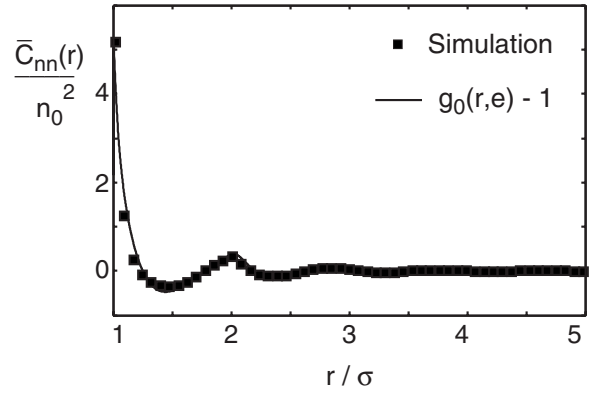

FIG. 3. The density correlation function $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.50$ with $L=32.5\sigma$ and $e=0.90$ as a function of the distance r. The solid line expresses $g_0(r,e)$ obtained by Lutsko [6].

shear rate to $\bar{C}_{nn}(r)$ because of its simplicity, the agreement between the results of our simulation and our theory seems to be perfect. Thus, it is hard to find any contributions of the shear in the short-range structure of the density correlation function, as is known in dense elastic liquids.

However, the above results do not mean that contributions of the shear to the density correlation function are not important. Indeed, we find the existence of a power-law tail which might be consistent with the theoretical prediction $\bar{C}_{nn}(r) \sim r^{-11/3}$ in Eq. (63) for $r \gg \sigma$ (see Fig. 4 for $\phi=0.185$, and $\dot{\gamma}=0.92\tau_0^{-1}$ and $1.84\tau_0^{-1}$). It should be noted that the shear rate dependence of $\bar{C}_{nn}(r)$ cannot be observed, because the scaled shear rate $\dot{\gamma}^* \sim \dot{\gamma}/\sqrt{T} \sim \sqrt{1-e^2}$ in Eq. (56) is independent of $\dot{\gamma}$ in the steady state. The range of the tail obeying a power law is not wide enough to verify the theoretical prediction in Fig. 4 because of the large statistical errors. However, as will be shown in Figs. 7 and 8, the numerical data for $\bar{C}_{pp}(r)$ are consistent with the theoretical prediction. To
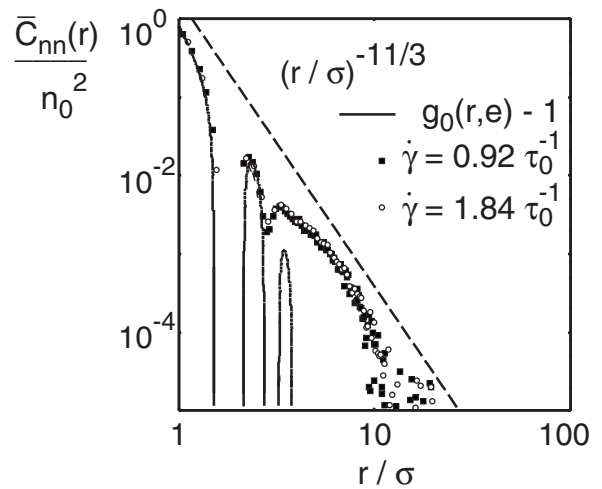

FIG. 4. The double-log plot of $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.185$ with $L=89\sigma$, and $e=0.83$ as a function of the distance r. Here, the solid line is that for the homogeneous case obtained by Lutsko [6], and the plotted data are obtained in the case of $\dot{\gamma}=0.92\tau_0^{-1}$ and $1.84\tau_0^{-1}$.
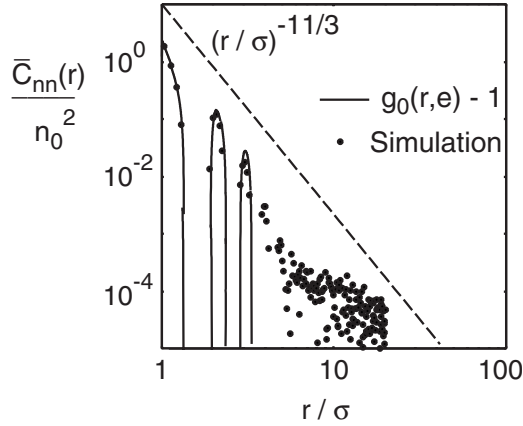
FIG. 5. The double-log plot of $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.37$ with $L=72\sigma$, $e=0.90$ as a function of the distance r. Here, the solid line is that for the homogeneous case obtained by Lutsko [6].

confirm the quantitative accuracy of our theory in $\bar{C}_{nn}(r)$, we need more extensive simulations to reduce the statistical errors.

Figures 5 and 6 show the behavior of $\bar{C}_{nn}(r)$ for $\phi=0.37$ and 0.50, respectively. We use $e=0.90$ and $L=72\sigma$ for $\phi=0.37$, and $e=0.90$ and $L=32.5\sigma$ for $\phi=0.50$. Although we can identify the existence of a power-law tail, it is hard to verify whether the tail satisfies $r^{-11/3}$ for $\phi=0.37$ and 0.50 because of the large statistical errors.

The long-range momentum correlation function that satisfies $r^{-5/3}$ can be observed more clearly than the case of the density correlation function as shown in Figs. 7 and 8. Figure 7 is the result of our simulation for $\phi=0.37$ and $e=0.90$. Figure 8 shows the result for $\phi=0.50$ and $e=0.90$. Although there is an apparent finite-size effect, $\bar{C}_{pp}(r)$ can be a universal function of r/L, and decays faster than the power-law function for $r>0.4L$ (see Figs. 7 and 8). We will have to check whether the oscillation of $\bar{C}_{pp}(r)$ in small r can be understood by our theory. To obtain the complete forms of
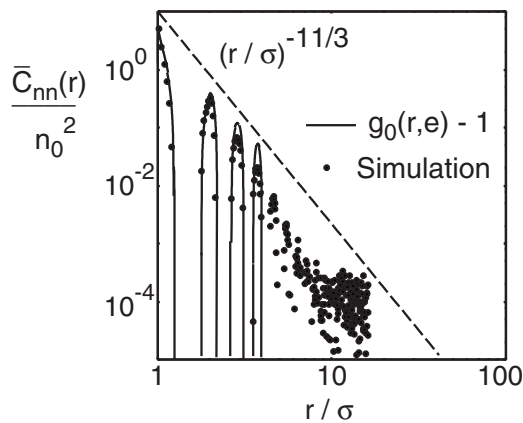
$\bar{C}_{nn}(r)$ and $\bar{C}_{pp}(r)$, we need to solve the eigenvalue problem of the inelastic Enskog operator and determine $\nu_1^*(k,e)$ and $\nu_2^*(k,e)$ explicitly. This will be our future work.

In this paper, we focus mainly on the results of sheared granular liquids, because granular liquids are more nontrivial than sheared isothermal elastic liquids, in which particles collide with each other without any loss of energy. We also perform the molecular-dynamics simulation for sheared elastic liquids with the velocity scaling thermostat to keep a constant temperature. Figures 9 and 10 are the results of the momentum correlation function of sheared elastic liquids with $\phi=0.093$ and 0.37, respectively. These results also support the power law $\bar{C}_{pp}(r)\sim r^{-5/3}$ even in the elastic case, although $\bar{C}_{pp}(r)$ has the negative value for $\phi=0.37$ due to the backscattering effect. Although the existence of a power law $\bar{C}_{pp}(r)\sim r^{-5/3}$ has been believed [28], this is the first time, to our knowledge, that the existence of such a tail has been verified from the molecular-dynamics simulation.

VI. DISCUSSION AND CONCLUSION

Now let us discuss our result. We should discuss the validity of the generalized fluctuating hydrodynamics. (i) Can
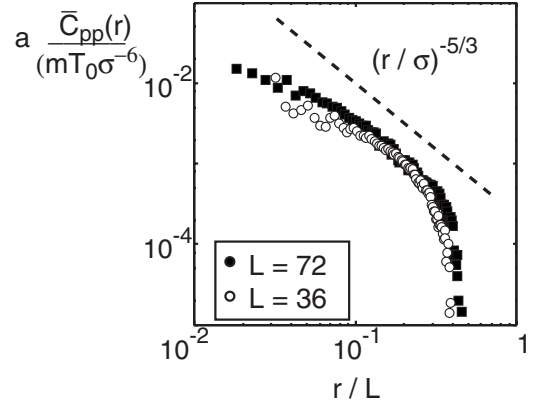

FIG. 7. The double-log plot of $\bar{C}_{pp}(r)$ for the volume fraction $\phi=0.37$ with $L/\sigma=36$, 72 and $e=0.90$ as a function of the distance r. Here, a is a fitting parameter, which is 0.4 for $L=36\sigma$ or 1.0 for $L=72\sigma$.

FIG. 6. The double-log plot of $\bar{C}_{nn}(r)$ for the volume fraction $\phi=0.50$ with $L=32.5\sigma$, $e=0.90$ as a function of the distance r. Here, the solid line is that for the homogeneous case obtained by Lutsko [6].
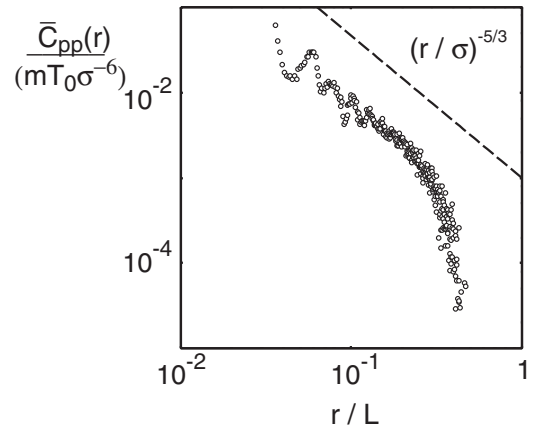

FIG. 8. The double-log plot of $\bar{C}_{pp}(r)$ for the volume fraction $\phi=0.50$ with $L=32.5\sigma$ and $e=0.90$ as a function of the distance r.
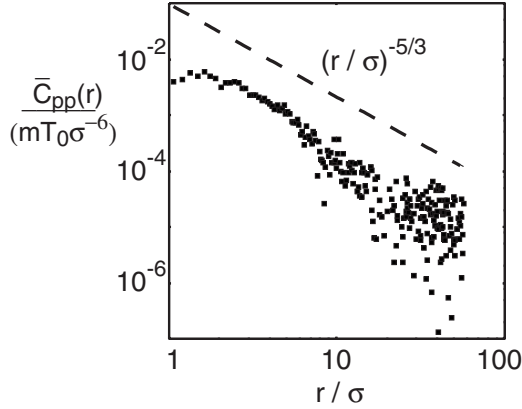
FIG. 9. The double-log plot of $\bar{C}_{pp}(r)$ for the volume fraction $\phi=0.093$ with $L=112\sigma$ and $e=1.0$ at $t=40\tau_0$ as a function of the distance r.

the noise be Gaussian? The answer is no. However, the non-Gaussian nature may not be important in our framework, because we only discuss the two-point correlation functions. When we discuss higher correlation functions, our theoretical treatment should be insufficient. (ii) What are contributions of the energy fluctuations? It might be important to include the effects of the temperature fluctuation for denser cases. When we take into account such fluctuations, we must solve a cubic equation for the eigenvalues satisfying Eq. (18) to obtain the explicit form of $\tilde{C}_{ii}(\boldsymbol{k})$. The analytic expression of the solution is, however, rather complicated. Hence, we do not take into account the fluctuations of the kinetic temperature. This treatment may be justified as long as the USF is stable. Indeed, the temperature is immediately relaxed to be uniform. (iii) The contribution of inelastic collisions also appears through the inelastic Enskog operator for granular gases. At present, we have not solved the eigenvalue problem of the inelastic Enskog operator. In this sense, our treatment in this paper is far from complete. The complete treatment will be discussed elsewhere.

Serious contributions of inelastic collisions appear in large systems, because USF is unstable for larger and highly inelastic systems. It should be noted that the shear flow is induced by the Lees-Edwards boundary condition. Using this boundary condition, USF is actually realized as shown in
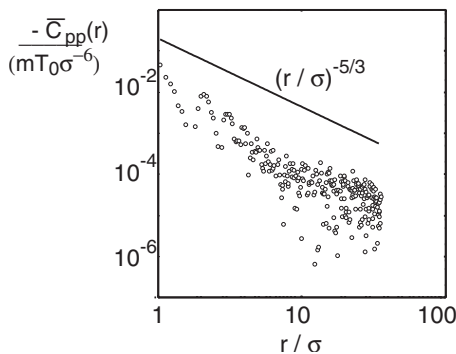

FIG. 10. The double-log plot of $\bar{C}_{pp}(r)$ for the volume fraction $\phi=0.37$ with $L=72\sigma$ and $e=1.0$ at $t=30\tau_0$ as a function of the distance r.
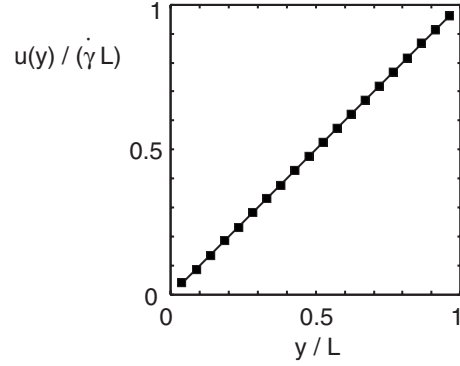

FIG. 11. The profile of the velocity $u(y)$ along the flow direction as a function of the position y for $\phi=0.37$, $e=0.90$, and $L=32\sigma$ with $t=20\tau_0$.

Fig. 11, which is in contrast to the shear flow obtained by moving a physical wall [13]. However, our long-time simulation for larger systems deviates from the theoretical prediction. In such a system, the linearized generalized hydrodynamics cannot be used, and the power-law correlation function may disappear. This nonstationary tendency expected from unstable USF in larger systems can be observed in our simulation. Figure 12 shows the time evolution of $\bar{C}_{pp}(r)$ for the system sizes $L=36\sigma$ and 72σ in the case of $e=0.90$ and $\phi=0.37$. It is clear that the result of a smaller system ($L=36\sigma$) converges, but the result of a larger system ($L=72\sigma$) does not converge. Although we have not identified the reason why we could not obtain the converged results in larger systems, it is likely to take place of the evolution of nonuniform structure [13,51–53].

It should be noted that the instability of USF can be reduced under gravity, which is not included in this paper [5,10]. Hence, we expect that the predicted long-range correlation can be observed in experiments for sheared granular materials [18], where the stable shear flow is realized.

In this paper, we demonstrate that the framework based on the generalized hydrodynamics is useful to determine the structure factor or the pair-correlation function. This result is significant in the following sense. First, we clarify the significant contribution of the shear to the structure factor of dense isothermal liquids, which is usually ignored in the pa-
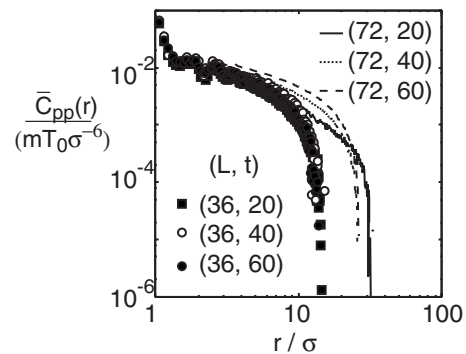

FIG. 12. The time dependence of $\bar{C}_{pp}(r)$ for the volume fraction $\phi=0.37$ at $t=20\tau_0, 40\tau_0, 60\tau_0$ for $L=36\sigma$ and 72σ in the case of $e=0.90$.

pers to discuss glassy behaviors of sheared liquids [54,55]. Second, the determination of the structure factor gives complementary information to MCT for sheared granular liquids [17].

In conclusion, we apply the generalized fluctuating hydrodynamics to isothermal sheared moderate dense liquids for both elastic cases and inelastic cases. The theory predicts that the density correlation function and the momentum correlation function, respectively, obey power laws $r^{-11/3}$ and $r^{-5/3}$. Our theory can be valid even for the short-range scale for $\sigma \leq r \leq 5\sigma$. The density correlation function for the short-range scale can be approximated by the theory introduced by Lutsko [6]. These results have been verified through a comparison with the simulation.

ACKNOWLEDGMENTS

APPENDIX A: APPROXIMATE EXPRESSION OF $g_0(r,e)$

In this appendix, we show the outline of the unsheared pair-correlation function obtained by Lutsko [6], who developed the generalized mean spherical approximation in describing a homogeneous cooling process of granular liquids.

The approximate expression of $g_0(e,r)$ is obtained from the inverse Laplace transform of the function $G(t)$ satisfying

$$G(t) = \int_0^\infty dr e^{-tr} r g_0(r,e), \tag{A1}$$

where $G(t)$ is given by

$$G(t) = \frac{tF(t)e^{-t}}{1 + 12\phi F(t)e^{-t}} \tag{A2}$$

with the volume fraction $\phi \equiv n_0 \sigma^3 \pi/6$. Here, we introduce $F(t)$ as

$$F(t) = \frac{-(1 + A_1 t + A_2 t^2)}{12\phi(S_0 + S_1 t + S_2 t^2 + S_3 t^3 + S_4 t^4)}, \tag{A3}$$

where $S_0 = 1$, $S_1 = A_1 - 1$, $S_2 = A_2 - A_1 + 1/2$, $S_3 = -A_2 + A_1/2 - (1 + 2\phi)/(12\phi)$, and $S_4 = A_2/2 - (1 + 2\phi)/(12\phi)A_1 + (2 + \phi)/(24\phi)$ with

$$A_1 = \frac{1}{2} + \sqrt{\frac{(\phi - 1)^2 - [6\phi g_0(\sigma,e) + 1]Z}{12\phi\{(2 + \phi) - 2g_0(\sigma,e)(\phi - 1)^2\}}},$$

$$A_2 = g_0(\sigma,e)\frac{(1 + 2\phi)A_1 - (2 + \phi)/2}{1 + 6\phi g_0(\sigma,e)}. \tag{A4}$$

In Eq. (A4), the pair-correlation function $g_0(\sigma,e)$ at contact is given by the equilibrium pair-correlation function at con-

tact $g_{eq}(\sigma) = (1 - \phi/2)/(1 - \phi)^3$ as $g_0(\sigma,e) = (1 + e)g_{eq}(\sigma)/(2e)$. In addition, Z is given as

$$Z = \left(1 + \frac{1 + e}{2}(Z_{eq}^{-1} - 1)\right)^{-1}, \tag{A5}$$

where

$$Z_{eq} = \left(\frac{\partial}{\partial n}\bar{P}\right)^{-1}, \tag{A6}$$

with $\bar{P} \equiv n[1 + 4\phi g_{eq}(\sigma)]/m$.

APPENDIX B: TRANSFORMATION BETWEEN CARTESIAN COORDINATE AND OBLIQUE COORDINATE

In the calculation of Eqs. (24)–(39), (46), and (52), it is convenient to use the oblique coordinate, where the vector $\hat{\mathbf{z}}$ of Eq. (9) in the Cartesian coordinate is transformed to the vector $\bar{\mathbf{z}}$ in the oblique coordinate as

$$\hat{z} = \mathsf{T}\bar{z}, \tag{B1}$$

where the matrix T is given by

$$\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & k_x/k & -k_y k_x/(kk_\perp) & k_z/k_\perp \\ 0 & k_y/k & k_\perp/k & 0 \\ 0 & k_z/k & -k_y k_z/(kk_\perp) & -k_x/k_\perp \end{bmatrix}. \tag{B2}$$

Here, we note that the second, third, and fourth components of $\bar{\mathbf{z}}$ are given by $\bar{z}_2 = \delta u_\parallel$, $\bar{z}_3 = \delta u_t$, and $\bar{z}_4 = \delta u_s$, where $\delta u_\parallel = \mathbf{e}_\parallel \cdot \delta\mathbf{u}$, $\delta u_t = \mathbf{e}_t \cdot \delta\mathbf{u}$, $\delta u_s = \mathbf{e}_s \cdot \delta\mathbf{u}$, and

$$\mathbf{u} = \delta u_\parallel \mathbf{e}_\parallel + \delta u_t \mathbf{e}_t + \delta u_s \mathbf{e}_s \tag{B3}$$

with

$$\mathbf{e}_\parallel^T = (k_x/k, k_y/k, k_z/k),$$

$$\mathbf{e}_t^T = (-k_y k_x/(kk_\perp), k_\perp/k, -k_y k_z/(kk_\perp)),$$

$$\mathbf{e}_s^T = (k_z/k_\perp, 0, -k_x/k_\perp).$$

From this transformation, the right eigenvectors and the left eigenvectors in the oblique coordinate are given, respectively, by

$$\bar{\psi}^{(1)T} = \frac{1}{N_+}(ikn_0\sigma^3, \lambda_+, 0, 0), \tag{B4}$$

$$\bar{\psi}^{(2)T} = \frac{1}{N_-}(ikn_0\sigma^3, \lambda_-, 0, 0), \tag{B5}$$

$$\bar{\psi}^{(3)T} = (0, 0, 1, M(\mathbf{k})), \tag{B6}$$

$$\bar{\psi}^{(4)T} = (0, 0, 0, 1), \tag{B7}$$

and

$$\bar{\varphi}^{(1)} = \frac{1}{N_+}(ikp^*(k,e), \lambda_+, 0, 0), \tag{B8}$$

$$\overline{\boldsymbol{\varphi}}^{(2)} = \frac{1}{N_-}(ikp^*(k,e), \lambda_-, 0, 0), \tag{B9}$$

$$\overline{\boldsymbol{\varphi}}^{(3)} = (0,0,1,0), \tag{B10}$$

$$\overline{\boldsymbol{\varphi}}^{(4)} = (0,0,-M(\boldsymbol{k}),1). \tag{B11}$$

APPENDIX C: EXPLICIT REPRESENTATION OF $\widetilde{C}_{nn}(\boldsymbol{k})$

In this appendix, we explicitly derive Eq. (51). $\widetilde{C}_{nn}(\boldsymbol{k})$ is related to $\widetilde{C}_{11}(\boldsymbol{k})$ as $\widetilde{C}_{nn}(\boldsymbol{k}) = \sigma^{-3}\widetilde{C}_{11}(\boldsymbol{k})$. Using Eq. (44), we obtain

$$\widetilde{C}_{11}(\boldsymbol{k}) = \int_0^\infty d\bar{t} \sum_{l,m} \widetilde{\psi}_1^{(l)}(\boldsymbol{k},\bar{t}) \widetilde{\psi}_1^{(m)}(-\boldsymbol{k},\bar{t}) F^{(lm)}[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{t})] = 4An_0^2\sigma^6 \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s}\nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\{\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]+\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]\}}$$

$$\times \frac{k^2}{N_+(k)N_-(k)} \frac{\lambda_+[k(\dot{\gamma}^*\bar{s})]\lambda_-[k(\dot{\gamma}^*\bar{s})]\nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_+[k(\dot{\gamma}^*\bar{s})]N_-[k(\dot{\gamma}^*\bar{s})]} + 2An_0^2\sigma^6 \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[2k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]}$$

$$\times \frac{k^2}{N_+(k)^2} \frac{\lambda_+[k(\dot{\gamma}^*\bar{s})]^2\nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_+[k(\dot{\gamma}^*\bar{s})]^2} + 2An_0^2\sigma^6 \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[2k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]}$$

$$\times \frac{k^2}{N_-(k)^2} \frac{\lambda_-[k(\dot{\gamma}^*\bar{s})]^2\nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_-[k(\dot{\gamma}^*\bar{s})]^2}. \tag{C1}$$

From the relations

$$\frac{d}{d\bar{t}} e^{-\int_0^{\bar{t}} d\bar{s}\nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} = -\nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{t})]\widetilde{k}(\dot{\gamma}^*\bar{t})^2 e^{-\int_0^{\bar{t}} d\bar{s}\nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2}, \tag{C2}$$

$$\frac{d}{d\bar{t}} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} = -\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{t})] e^{-\int_0^{\bar{t}} d\bar{s}\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]}, \tag{C3}$$

$$\frac{d}{d\bar{t}} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} = -\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{t})] e^{-\int_0^{\bar{t}} d\bar{s}\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]}, \tag{C4}$$

Eq. (C1) can be rewritten as

$$\widetilde{C}_{11}(\boldsymbol{k}) = An_0^2\sigma^6 \left\{ \frac{4k^2\lambda_+(k)\lambda_-(k)}{N_+(k)^2N_-(k)^2} + \frac{k^4\lambda_+(k)\nu_1^*(k,e)}{N_+(k)^4} + \frac{k^4\lambda_-(k)\nu_1^*(k,e)}{N_-(k)^4} \right\}$$

$$+ An_0^2\sigma^6 \int_0^\infty d\bar{t} \frac{4k^2\lambda_+(k)\lambda_-(k)}{N_+(k)N_-(k)} e^{-\int_0^{\bar{t}} d\bar{s}\nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} \frac{d}{dt} \{ e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\{\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]+\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]\}} G_{12}[k(\dot{\gamma}^*\bar{t})] \}$$

$$+ An_0^2\sigma^6 \int_0^\infty d\bar{t} \frac{k^4\lambda_+(k)\nu_1^*(k,e)}{N_+(k)^2} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} \frac{d}{dt} \{ e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[2k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]} G_{11}[k(\dot{\gamma}^*\bar{t})] \}$$

$$+ An_0^2\sigma^6 \int_0^\infty d\bar{t} \frac{k^4\lambda_-(k)\nu_1^*(k,e)}{N_-(k)^2} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} \frac{d}{dt} \{ e^{-\int_0^{\bar{t}} d\bar{s}[2k_x\widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]} G_{22}[k(\dot{\gamma}^*\bar{t})] \}, \tag{C5}$$

where we have introduced

$$G_{12}(k) = \frac{\lambda_+(k)\lambda_-(k)}{N_+(k)N_1(k)}, \tag{C6}$$

$$G_{11}(k) = \frac{\lambda_+(k)\nu_1^*(k,e)k^2}{N_+(k)^2}, \tag{C7}$$

$$G_{22}(k) = \frac{\lambda_-(k)\nu_1^*(k,e)k^2}{N_-(k)^2}. \tag{C8}$$

It is possible to obtain the simpler expression for $\widetilde{C}_{11}(\boldsymbol{k})$ from the relation $[4k^2\lambda_+\lambda_-/(N_+^2N_-^2) + \lambda_+\nu_1^*(k,e)k^4/N_+^4 + \lambda_-\nu_1^*(k,e)k^4/N_-^4] = [\sigma^3 n_0 p^*(k,e)]^{-1} = S_0(k,e)/(\sigma^3 n_0 A)$, and

$$\frac{d}{dt}\{e^{-\int_0^{\bar{t}}d\bar{s}\dot{\gamma}^*[k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\{\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]+\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]\}}G_{12}[k(\dot{\gamma}^*\bar{t})]\}$$

$$=\dot{\gamma}^*e^{-\int_0^{\bar{t}}d\bar{s}\dot{\gamma}^*[k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\{\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]+\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]\}}H_{12}[\tilde{k}(\dot{\gamma}^*\bar{t})],$$

$$(\text{C9})$$

$$\frac{d}{dt}\{e^{-\int_0^{\bar{t}}d\bar{s}\dot{\gamma}^*[2k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]}G_{11}[k(\dot{\gamma}^*\bar{t})]\}$$

$$=\dot{\gamma}^*e^{-\int_0^{\bar{t}}d\bar{s}\dot{\gamma}^*[2k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]}H_{11}[\tilde{k}(\dot{\gamma}^*\bar{t})],$$

$$(\text{C10})$$

$$\frac{d}{dt}\{e^{-\int_0^{\bar{t}}d\bar{s}\dot{\gamma}^*[2k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]}G_{22}[k(\dot{\gamma}^*\bar{t})]\}$$

$$=\dot{\gamma}^*e^{-\int_0^{\bar{t}}d\bar{s}[2k_x\tilde{k}_y(\dot{\gamma}^*\bar{s})/\tilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]}H_{22}[\tilde{k}(\dot{\gamma}^*\bar{t})],$$

$$(\text{C11})$$

with

$$H_{12}(k)=k\frac{dG_{12}(k)}{dk}-[\xi^{(1)}(k)+\xi^{(2)}(k)]G_{12}(k), \quad (\text{C12})$$

$$H_{11}(k)=k\frac{dG_{11}(k)}{dk}-2\xi^{(1)}(k)G_{11}(k), \quad (\text{C13})$$

$$H_{22}(k)=k\frac{dG_{22}(k)}{dk}-2\xi^{(2)}(k)G_{22}(k). \quad (\text{C14})$$

Substituting Eqs. (C9)–(C14) into Eq. (C5), we obtain

$$\tilde{C}_{11}(\boldsymbol{k})=n_0\sigma^3\{S_0(k,e)+\dot{\gamma}^*\tilde{\Delta}_1(\boldsymbol{k})\}, \quad (\text{C15})$$

where we have introduced

$$\tilde{\Delta}_1(\boldsymbol{k})=An_0\sigma^3\int_0^\infty d\bar{t}\frac{4k^2H_{12}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)N_-(k)}\frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}e^{-\int_0^{\bar{t}}\bar{s}\{\lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})]+\lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]\}}$$

$$+An_0\sigma^3\int_0^\infty d\bar{t}\frac{k^2H_{11}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)^2}\frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}e^{-2\int_0^{\bar{t}}d\bar{s}\lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}+An_0\sigma^3\int_0^\infty d\bar{t}\frac{k^2H_{22}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_-(k)^2}\frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*t)^2}e^{-2\int_0^{\bar{t}}d\bar{s}\lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}.$$

$$(\text{C16})$$

We thus obtain Eq. (52) from Eq. (C15) with $\tilde{C}_{nn}(\boldsymbol{k},\bar{t})=\sigma^{-3}\tilde{C}_{11}(\boldsymbol{k},\bar{t})$.

APPENDIX D: EXPLICIT REPRESENTATION OF $\tilde{C}_{pp}(\boldsymbol{k})$

In this appendix, we also explicitly derive Eq. (52). The momentum correlation $\tilde{C}_{pp}(\boldsymbol{k})$ can be written as

$$\tilde{C}_{pp}(\boldsymbol{k})=(mn_0\sigma/t_E)^2\{\tilde{C}_{22}(\boldsymbol{k})+\tilde{C}_{33}(\boldsymbol{k})+\tilde{C}_{44}(\boldsymbol{k})\}, \quad (\text{D1})$$

where

$$\tilde{C}_{22}(\boldsymbol{k})+\tilde{C}_{33}(\boldsymbol{k})+\tilde{C}_{44}(\boldsymbol{k})$$

$$=\int_0^\infty d\bar{t}\sum_{l,m}\sum_{\alpha=2,3,4}\tilde{\psi}_\alpha^{(l)}(\boldsymbol{k},\bar{t})\tilde{\psi}_\alpha^{(m)}(-\boldsymbol{k},\bar{t})F^{(lm)}[\tilde{k}(\dot{\gamma}^*\bar{t})]$$

$$=\int_0^\infty d\bar{t}\sum_{l,m}\sum_{\alpha=2,3,4}(-1)^{\alpha+1}\bar{\psi}_\alpha^{(l)}(\boldsymbol{k},\bar{t})\bar{\psi}_\alpha^{(m)}(-\boldsymbol{k},\bar{t})F^{(lm)}$$

$$\times[\tilde{k}(\dot{\gamma}^*\bar{t})]. \quad (\text{D2})$$

Here, to obtain Eq. (D2) we have used the relation

$$\tilde{\boldsymbol{\psi}}^{(l)}(\boldsymbol{k},\bar{t})=\mathsf{T}\cdot\bar{\boldsymbol{\psi}}^{(l)}(\boldsymbol{k},\bar{t})$$

and

$$\Sigma_{\alpha=2,3,4}\mathsf{T}_{\alpha\beta}(\boldsymbol{k})\mathsf{T}_{\alpha\gamma}(-\boldsymbol{k})=(-1)^\gamma\Sigma_{\alpha=2,3,4}\mathsf{T}_{\beta\alpha}^{-1}(\boldsymbol{k})\mathsf{T}_{\alpha\gamma}(\boldsymbol{k})$$

$$=(-1)^{\gamma+1}\delta_{\beta\gamma}.$$

Introducing

$$\tilde{D}_\alpha(\boldsymbol{k})\equiv\int_0^\infty d\bar{t}\sum_{l,m}\bar{\psi}_\alpha^{(l)}(\boldsymbol{k},\bar{t})\bar{\psi}_\alpha^{(m)}(-\boldsymbol{k},\bar{t})F^{(lm)}[\tilde{k}(\dot{\gamma}^*\bar{t})],$$

$$(\text{D3})$$

we can rewrite $\tilde{C}_{pp}(\boldsymbol{k})$ as

$$\tilde{C}_{pp}(\boldsymbol{k})=(mn_0\sigma/t_E)^2\{-\tilde{D}_2(\boldsymbol{k})+\tilde{D}_3(\boldsymbol{k})-\tilde{D}_4(\boldsymbol{k})\}. \quad (\text{D4})$$

From Eq. (D3), $\tilde{D}_2(\boldsymbol{k})$ can be rewritten as

$$\widetilde{D}_2(\boldsymbol{k}) = 4A \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s} \nu_1^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} e^{-\int_0^{\bar{t}} d\bar{s} \dot{\gamma}^*[k_x \widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]} \{\xi^{(1)}[k(\dot{\gamma}^*\bar{s})] + \xi^{(2)}[k(\dot{\gamma}^*\bar{s})]\}$$

$$\times \frac{\lambda_+(k)\lambda_-(k)}{N_+(k)N_-(k)} \frac{\lambda_+[k(\dot{\gamma}^*\bar{s})]\lambda_-[k(\dot{\gamma}^*\bar{s})]\nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_+[k(\dot{\gamma}^*\bar{s})]N_-[k(\dot{\gamma}^*\bar{s})]}$$

$$+ 2A \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_+^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[2k_x \widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(1)}[k(\dot{\gamma}^*\bar{s})]} \frac{\lambda_+(k)^2}{N_+(k)^2} \frac{\lambda_+[k(\dot{\gamma}^*\bar{s})]^2 \nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_+[k(\dot{\gamma}^*\bar{s})]^2}$$

$$+ 2A \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s}\lambda_-^*[\widetilde{\boldsymbol{k}}(\dot{\gamma}^*\bar{s})]} e^{-\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[2k_x \widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]\xi^{(2)}[k(\dot{\gamma}^*\bar{s})]} \frac{\lambda_-(k)^2}{N_-(k)^2} \frac{\lambda_-[k(\dot{\gamma}^*\bar{s})]^2 \nu_1^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2}{N_-[k(\dot{\gamma}^*\bar{s})]^2}. \quad \text{(D5)}$$

From the parallel procedure to that in Appendix C, we obtain

$$\widetilde{D}_2(\boldsymbol{k}) = -A \left\{ \frac{4\lambda_+(k)^2\lambda_-(k)^2}{N_+(k)^2 N_-(k)^2} + \frac{\lambda_+(k)^4}{N_+(k)^4} + \frac{\lambda_-(k)^4}{N_-(k)^4} \right\} - A\dot{\gamma}^* \widetilde{\Delta}_2(\boldsymbol{k}), \quad \text{(D6)}$$

where

$$\widetilde{\Delta}_2(\boldsymbol{k}) = \int_0^\infty d\bar{t} \frac{4\lambda_+(k)\lambda_-(k)H_{12}[\widetilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)N_-(k)} \frac{k_x \widetilde{k}_y(\dot{\gamma}^*\bar{t})}{\widetilde{k}(\dot{\gamma}^*\bar{t})^2} e^{-\int_0^{\bar{t}} d\bar{s}\{\lambda^{(1)}[\widetilde{k}(\dot{\gamma}^*\bar{s})]+\lambda^{(2)}[\widetilde{k}(\dot{\gamma}^*\bar{s})]\}}$$

$$+ \int_0^\infty d\bar{t} \frac{\lambda_+(k)^2 H_{11}[\widetilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)^2} \frac{k_x \widetilde{k}_y(\dot{\gamma}^*\bar{t})}{\widetilde{k}(\dot{\gamma}^*\bar{t})^2} e^{-2\int_0^{\bar{t}} d\bar{s}\lambda^{(1)}[\widetilde{k}(\dot{\gamma}^*\bar{s})]} + \int_0^\infty d\bar{t} \frac{\lambda_1(k)^2 H_{22}[\widetilde{k}(\dot{\gamma}^*\bar{t})]}{N_-(k)^2} \frac{k_x \widetilde{k}_y(\dot{\gamma}^*\bar{t})}{\widetilde{k}(\dot{\gamma}^*\bar{t})^2} e^{-2\int_0^{\bar{t}} d\bar{s}\lambda^{(2)}[\widetilde{k}(\dot{\gamma}^*\bar{s})]}. \quad \text{(D7)}$$

From $\quad 4\lambda_+(k)^2\lambda_-(k)^2/[N_+(k)^2 N_-(k)^2] + \lambda_+(k)^4/N_+(k)^4 + \lambda_-(k)^4/N_-(k)^4 = 1$, we find that $\widetilde{D}_2(\boldsymbol{k})$ is represented by Eq. (53).

From Eq. (D3), we obtain

$$\widetilde{D}_3(\boldsymbol{k}) = 2A \int_0^\infty d\bar{t} e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} e^{\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^* 2[k_x \widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]}$$

$$\times \nu_2^*[k(\dot{\gamma}^*\bar{s})]k(\dot{\gamma}^*\bar{s})^2. \quad \text{(D8)}$$

From the identities

$$\frac{d}{d\bar{t}} e^{-\int_0^{\bar{t}} d\bar{s}\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2}$$

$$= -\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{t})]\widetilde{k}(\dot{\gamma}^*\bar{t})^2 e^{-\int_0^{\bar{t}} d\bar{s}\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} \quad \text{(D9)}$$

and

$$e^{\int_0^{\bar{t}} d\bar{s}\dot{\gamma}^*[k_x \widetilde{k}_y(\dot{\gamma}^*\bar{s})/\widetilde{k}(\dot{\gamma}^*\bar{s})^2]} = \frac{\widetilde{k}(\dot{\gamma}^*\bar{s})}{k^2}, \quad \text{(D10)}$$

we find that $\widetilde{D}_3(\boldsymbol{k})$ is expressed as Eq. (54) with

$$\widetilde{\Delta}_3(\boldsymbol{k}) = \int_0^\infty d\bar{t} \frac{k_x \widetilde{k}_y(\dot{\gamma}^*\bar{t})}{k^2} e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2}. \quad \text{(D11)}$$

From Eq. (D3), we obtain

$$\widetilde{D}_4(\boldsymbol{k}) = -2A \int_0^\infty d\bar{t} \left\{ M^2(\boldsymbol{k}) \frac{\widetilde{k}(\dot{\gamma}^*\bar{s})^2}{k^2} \right.$$

$$- 2M(\boldsymbol{k})M[\widetilde{k}(\dot{\gamma}^*\bar{s})] \frac{\widetilde{k}(\dot{\gamma}^*\bar{s})}{k} + M[\widetilde{k}(\dot{\gamma}^*\bar{s})]^2 + 1 \left. \right\}$$

$$\times e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2} \nu_2^*[k(\dot{\gamma}^*\bar{t})]k(\dot{\gamma}^*\bar{t})^2. \quad \text{(D12)}$$

With the aid of Eq. (D9), we confirm that $\widetilde{D}_4(\boldsymbol{k})$ can be represented by Eq. (55), where

$$\widetilde{\Delta}_4(\boldsymbol{k}) = \int_0^\infty d\bar{t} \left\{ \frac{k_x \widetilde{k}_y(\dot{\gamma}^*\bar{t})}{\widetilde{k}(\dot{\gamma}^*\bar{t})^2} F(\boldsymbol{k},\bar{t}) \right.$$

$$\left. - \frac{k_z}{\widetilde{k}(\dot{\gamma}^*\bar{t})} \right\} F(\boldsymbol{k},\bar{t}) e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\widetilde{k}(\dot{\gamma}^*\bar{s})]\widetilde{k}(\dot{\gamma}^*\bar{s})^2}, \quad \text{(D13)}$$

and $F(\boldsymbol{k},t) = M[\hat{\boldsymbol{k}}(t)] - \hat{k}(t)M(k)/k$.

Finally, substituting eqs. (53)–(55) into Eq. (D1) with Eq. (D2), we obtain the explicit representation of $\widetilde{C}_{pp}(\boldsymbol{k})$ as Eq. (52).

APPENDIX E: EVALUATION OF THE ASYMPTOTIC FORM OF $\bar{\Delta}_j(r)$

In this appendix, we evaluate the asymptotic form of $\bar{\Delta}_j(r)$ for $r \gg l_c$.

1. Asymptotic form of $\bar{\Delta}_1(r)$

We note that $\bar{\Delta}_1(r)$ can be separated into three parts,

$$\bar{\Delta}_1(r) = \bar{\Delta}_{11}(r) + \bar{\Delta}_{12}(r) + \bar{\Delta}_{13}(r), \tag{E1}$$

where

$$\bar{\Delta}_{11}(r) = \frac{A}{n_0\sigma^6} \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$
$$\times \int_0^\infty d\bar{t} \frac{4k^2 H_{12}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)N_-(k)} \frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}$$
$$\times e^{-\int_0^{\bar{t}}\bar{s}\{\lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})]+\lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]\}}, \tag{E2}$$

$$\bar{\Delta}_{12}(r) = \frac{A}{n_0\sigma^6} \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$
$$\times \int_0^\infty d\bar{t} \frac{k^2 H_{11}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)^2} \frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}$$
$$\times e^{-2\int_0^{\bar{t}}d\bar{s}\lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}, \tag{E3}$$

$$\bar{\Delta}_{13}(r) = \frac{A}{n_0\sigma^6} \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$
$$\times \int_0^\infty d\bar{t} \frac{k^2 H_{22}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_-(k)^2} \frac{k_x\tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*t)^2}$$
$$\times e^{-2\int_0^{\bar{t}}d\bar{s}\lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}. \tag{E4}$$

Let us introduce $l_c \equiv \sigma/\sqrt{\dot{\gamma}^*}$, $r^* \equiv r/l_c$, and the transformation $\bar{t}=\tau/\dot{\gamma}^*$, $\bar{s}=\tau'/\dot{\gamma}^*$, $\boldsymbol{k}=\sqrt{\dot{\gamma}^*}\boldsymbol{K}/r^*=\sigma\boldsymbol{K}/r$, and $\tilde{k}(\tau)=\sqrt{\dot{\gamma}^*}\tilde{K}(\tau)/r^*$. By using the new variables, Eqs. (E2)–(E4) can be rewritten as

$$\bar{\Delta}_{11}(r) = r^{*-5}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty d\tau \frac{4K^2 H_{12}(\sqrt{\dot{\gamma}^*}\tilde{K}(\tau)/r^*)}{N_+(\sqrt{\dot{\gamma}^*}K/r^*)N_-(\sqrt{\dot{\gamma}^*}K/r^*)} \frac{K_x\tilde{K}_y(\tau)}{\tilde{K}(\tau)^2}$$
$$\times e^{-\int_0^\tau\tau'\dot{\gamma}^{*-1}\{\lambda^{(1)}[\sqrt{\dot{\gamma}^*}\tilde{K}(\tau')/r^*]+\lambda^{(2)}[\sqrt{\dot{\gamma}^*}\tilde{K}(\tau')/r^*]\}}, \tag{E5}$$

$$\bar{\Delta}_{12}(r) = r^{*-5}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty d\tau \frac{K^2 H_{11}[\sqrt{\dot{\gamma}^*}\tilde{K}(\tau)/r^*]}{N_+(\sqrt{\dot{\gamma}^*}K/r^*)^2} \frac{K_x\tilde{K}_y(\tau)}{\tilde{K}(\tau)^2}$$
$$\times e^{-2\int_0^\tau d\tau'\dot{\gamma}^{*-1}\lambda^{(1)}[\sqrt{\dot{\gamma}^*}\tilde{K}(\tau')/r^*]}, \tag{E6}$$

$$\bar{\Delta}_{13}(r) = r^{*-5}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty d\tau \frac{K^2 H_{22}[\sqrt{\dot{\gamma}^*}\tilde{K}(\tau)/r^*]}{N_-(\sqrt{\dot{\gamma}^*}K/r^*)^2} \frac{K_x\tilde{K}_y(\tau)}{\tilde{K}(\tau)^2}$$
$$\times e^{-2\int_0^\tau d\tau'\dot{\gamma}^{*-1}\lambda^{(2)}[\sqrt{\dot{\gamma}^*}\tilde{k}(\tau')/r^*]}. \tag{E7}$$

Introducing the transformations $\tau=r^{*2/3}Y$ and $\tau'=r^{*2/3}Y'$ to extract the asymptotic forms for $r^* \gg 1$, $\bar{\Delta}_{1j}(r)$ $(j=1,2,3)$ can be rewritten as

$$\bar{\Delta}_{11}(r) = r^{*-13/3}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty dY \frac{4K^2 H_{12}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*]}{N_+(\sqrt{\dot{\gamma}^*}K/r^*)N_-(\sqrt{\dot{\gamma}^*}K/r^*)} \frac{K_x\tilde{K}_y(r^{*2/3}Y)}{\tilde{K}(r^{*2/3}Y)^2}$$
$$\times e^{-\int_0^\tau dY'\dot{\gamma}^{*-1}r^{*2/3}\{\lambda^{(1)}[\sqrt{\dot{\gamma}^*}\boldsymbol{K}(r^{*2/3}Y')/r^*]+\lambda^{(2)}[\sqrt{\dot{\gamma}^*}\boldsymbol{K}(r^{*2/3}Y')/r^*]\}}, \tag{E8}$$

$$\bar{\Delta}_{12}(r) = r^{*-13/3}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty dY \frac{K^2 H_{11}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*]}{N_+(\sqrt{\dot{\gamma}^*}K/r^*)^2} \frac{K_x\tilde{K}_y(r^{*2/3}Y)}{\tilde{K}(r^{*2/3}y)^2}$$
$$\times e^{-2\int_0^\tau dY'\dot{\gamma}^{*-1}r^{*2/3}\lambda^{(1)}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y')/r^*]}, \tag{E9}$$

$$\bar{\Delta}_{13}(r) = r^{*-13/3}\frac{A}{n_0\sigma^3 l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$
$$\times \int_0^\infty dY \frac{K^2 H_{22}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*]}{N_-(\sqrt{\dot{\gamma}^*}K/r^*)^2} \frac{K_x\tilde{K}_y(r^{*2/3}Y)}{\tilde{K}(r^{*2/3}Y)^2}$$
$$\times e^{-2\int_0^\tau dY'\dot{\gamma}^{*-1}r^{*2/3}\lambda^{(2)}[\sqrt{\dot{\gamma}^*}\tilde{k}(r^{*2/3}Y')/r^*]}, \tag{E10}$$

where the functions included in Eqs. (E8)–(E10) have the asymptotic forms for $r^* \gg 1$,

$$\tilde{K}_y(\bar{r}^{2/3}Y) \simeq r^{*2/3}\{YK_x + O(r^{*-2/3})\}, \tag{E11}$$

$$\tilde{K}(r^{*2/3}Y')^2 \simeq r^{*4/3}\{Y'^2 K_x^2 + O(r^{*-2/3})\}, \tag{E12}$$

$$H_{12}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*] \simeq \frac{1}{2} + O(r^{*-2/3}), \tag{E13}$$

$$H_{11}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*] \simeq \frac{i\sqrt{\dot{\gamma}^*}\nu_1^*(0,t)}{2c_0}Y|K_x|r^{*-1/3} + O(r^{*-2/3}), \tag{E14}$$

$$H_{22}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*] \simeq -\frac{i\sqrt{\dot{\gamma}^*}\nu_1^*(0,t)}{2c_0}y|K_x|r^{*-1/3}$$
$$+ O(r^{*-2/3}), \tag{E15}$$

$$N_+(\sqrt{\dot{\gamma}^*}K/r^*) \simeq \sqrt{2\dot{\gamma}^*}Kic_0 r^{*-1} + O(r^{*-2}), \quad \text{(E16)}$$

$$N_-(\sqrt{\dot{\gamma}^*}K/r^*) \simeq \sqrt{2\dot{\gamma}^*}Kic_0 r^{*-1} + O(r^{*-2}), \quad \text{(E17)}$$

$$e^{-\int_0^\tau dY' \dot{\gamma}^{*-1} r^{*2/3} \lambda^{(1)}[\sqrt{\dot{\gamma}^*}K(r^{*2/3}Y')/r^*]}$$

$$\simeq e^{-ic_0 \dot{\gamma}^{*-1/2} \int_0^Y dy' |K_x| Y' r^{*1/2}} \sqrt{\frac{K}{|K_x| r^{*2/3} Y}}$$

$$\times e^{-\int_0^y dY' \nu_1^*(0,t) K_x^2 Y'^2} \{1 + O(r^{*-2/3})\}, \quad \text{(E18)}$$

$$e^{-\int_0^\tau dY' \dot{\gamma}^{*-1} r^{*2/3} \lambda^{(2)}[\sqrt{\dot{\gamma}^*}K(r^{*2/3}Y')/r^*]}$$

$$\simeq e^{ic_0 \dot{\gamma}^{*-1/2} \int_0^Y dY' |K_x| Y' r^{*1/2}} \sqrt{\frac{K}{|K_x| r^{*2/3} Y}}$$

$$\times e^{-\int_0^Y dY' \nu_1^*(0,t) K_x^2 Y'^2} \{1 + O(r^{*-2/3})\}, \quad \text{(E19)}$$

with $c_0^2 \equiv n_0 \sigma^3 p^*(0,e)$. Substituting these relations into Eqs. (E8)–(E10), we obtain

$$\bar{\Delta}_{11}(r) = J_{11} r^{*-11/3} + O(r^{*-13/3}),$$

$$|\bar{\Delta}_{12}(r)| = |\bar{\Delta}_{13}(r)| = J_{12} r^{*-4} + O(r^{*-14/3}), \quad \text{(E20)}$$

where we have introduced

$$J_{11} = \frac{A}{\dot{\gamma}^* c_0 n_0 \sigma^3 l_c^3} \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{1}{Y^2} \frac{K}{|K_x|} e^{-2\nu_1^*(0)Y^3/3},$$

$$\text{(E21)}$$

$$J_{12} = \frac{A}{4n_0 \sigma^3 l_c^3 \sqrt{\dot{\gamma}^*} c_0^3} \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{K}{Y} e^{-2\nu_1^*(0)Y^3/3}.$$

$$\text{(E22)}$$

Thus, the asymptotic form of $\bar{\Delta}_1(r)$ can be represented by Eq. (59).

2. Asymptotic form of $\bar{\Delta}_2(r)$

Similarly, $\bar{\Delta}_2(r)$ is given by summation of the three terms

$$\bar{\Delta}_2(r) = \bar{\Delta}_{21}(r) + \bar{\Delta}_{22}(r) + \bar{\Delta}_{23}(r), \quad \text{(E23)}$$

where

$$\bar{\Delta}_{21}(r) = \sigma^{-3} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$

$$\times \int_0^\infty d\bar{t} \frac{4\lambda_+(k)\lambda_-(k)H_{12}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)N_-(k)} \frac{k_x \tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}$$

$$\times e^{-\int_0^{\bar{t}} d\bar{s} \{\lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})] + \lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]\}}, \quad \text{(E24)}$$

$$\bar{\Delta}_{22}(r) = \sigma^{-3} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$

$$\times \int_0^\infty d\bar{t} \frac{\lambda_+(k)^2 H_{11}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_+(k)^2} \frac{k_x \tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}$$

$$\times e^{-2\int_0^{\bar{t}} d\bar{s} \lambda^{(1)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}, \quad \text{(E25)}$$

$$\bar{\Delta}_{23}(r) = \sigma^{-3} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$

$$\times \int_0^\infty d\bar{t} \frac{\lambda_1(k)^2 H_{22}[\tilde{k}(\dot{\gamma}^*\bar{t})]}{N_-(k)^2} \frac{k_x \tilde{k}_y(\dot{\gamma}^*\bar{t})}{\tilde{k}(\dot{\gamma}^*\bar{t})^2}$$

$$\times e^{-2\int_0^{\bar{t}} d\bar{s} \lambda^{(2)}[\tilde{k}(\dot{\gamma}^*\bar{s})]}. \quad \text{(E26)}$$

As in the case of the previous subsection, from the transformations $\bar{t} = r^{*2/3}Y/\dot{\gamma}^*$, $\bar{s} = r^{*2/3}Y'/\dot{\gamma}^*$, and $\mathbf{k} = \sqrt{\dot{\gamma}^*}\mathbf{K}/r^*$, we obtain

$$\bar{\Delta}_{21}(r) = \frac{r^{*-7/3}}{\dot{\gamma}^* l_c^3} \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{4\lambda_+(\sqrt{\dot{\gamma}^*}K/r^*)\lambda_-(\sqrt{\dot{\gamma}^*}K/r^*)H_{12}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*]}{N_+(\sqrt{\dot{\gamma}^*}K/r^*)N_-(\sqrt{\dot{\gamma}^*}K/r^*)}$$

$$\times \frac{K_x \tilde{K}_y(r^{*2/3}Y)}{\tilde{K}(r^{*2/3}Y)^2} e^{-\int_0^Y dY' \dot{\gamma}^{*-1} r^{*2/3} \{\lambda^{(1)}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y')/r^*] + \lambda^{(2)}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y')/r^*]\}}, \quad \text{(E27)}$$

$$\bar{\Delta}_{22}(r) = \frac{r^{*-7/3}}{\dot{\gamma}^* l_c^3} \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{\lambda_+(\sqrt{\dot{\gamma}^*}K/r^*)^2 H_{11}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y)/r^*]}{N_+(\sqrt{\dot{\gamma}^*}\mathbf{K}/r^*)^2} \frac{K_x \tilde{K}_y(r^{*2/3}Y)}{\tilde{K}(r^{*2/3}Y)^2}$$

$$\times e^{-2\int_0^Y dY' \dot{\gamma}^{*-1} r^{*2/3} \lambda^{(1)}[\sqrt{\dot{\gamma}^*}\tilde{K}(r^{*2/3}Y')/r^*]}, \quad \text{(E28)}$$

$$\bar{\Delta}_{23}(r) = \frac{r^{*-7/3}}{\dot{\gamma}^* l_c^3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{\lambda_-(\sqrt{\dot{\gamma}^*}K/r^*)^2 H_{22}[\sqrt{\dot{\gamma}^*}\widetilde{K}(r^{*2/3}Y)/r^*]}{N_-(\sqrt{\dot{\gamma}^*}K/r^*)^2} \frac{K_x \widetilde{K}_y(r^{*2/3}Y)}{\widetilde{K}(r^{*2/3}Y)^2}$$

$$\times e^{-2\int_0^Y dY' \dot{\gamma}^{*-1}\lambda^{(2)}r^{*2/3}[\sqrt{\dot{\gamma}^*}\widetilde{K}(r^{*2/3}Y')/r^*]}. \tag{E29}$$

From Eqs. (E11)–(E19) and Eqs. (E27)–(E29), we obtain

$$\bar{\Delta}_{21}(r) = \frac{c_0 n_0 \sigma^3}{A} J_{11} r^{*-11/3} + O(r^{*-13/3}), \tag{E30}$$

$$|\bar{\Delta}_{22}(r)| = |\bar{\Delta}_{23}(r)| = \frac{c_0 n_0 \sigma^3}{A} J_{12} r^{*-4} + O(r^{*-14/3}). \tag{E31}$$

We thus find the asymptotic form of $\bar{\Delta}_2(r)$ is given by Eq. (60).

3. Asymptotic Form of $\bar{\Delta}_3(r)$

$\bar{\Delta}_3(r)$ can be represented by

$$\bar{\Delta}_3(r) = \sigma^{-3} \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma} \int_0^\infty d\bar{t} \frac{k_x \widetilde{k}_y(\dot{\gamma}^* \bar{t})}{k^2}$$

$$\times e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\bar{k}(\dot{\gamma}^* \bar{s}),0]\widetilde{k}(\dot{\gamma}^* \bar{s})^2}. \tag{E32}$$

From the transformations $\bar{t} = \tau/\dot{\gamma}^*$, $\bar{s} = \tau'/\dot{\gamma}^*$, and $\boldsymbol{k} = \sqrt{\dot{\gamma}^*}K/\bar{r}$, we can rewritten Eq. (E32) as

$$\bar{\Delta}_3(r) = \dot{\gamma}^{*-1} l_c^{-3} r^{*-3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty d\tau \frac{K_x \widetilde{K}_y(\tau)}{K^2}$$

$$\times e^{-\int_0^\tau d\tau' 2\nu_2^*[\sqrt{\dot{\gamma}^*}\widetilde{K}(\tau)/r^*]\widetilde{K}(\tau)^2/r^{*2}}. \tag{E33}$$

Introducing the other transformations $\tau = r^{*2/3}y$ and $\tau' = r^{*2/3}y'$, we obtain

$$\bar{\Delta}_3(r) = \dot{\gamma}^{*-1} l_c^{-3} r^{*-7/3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dy \frac{K_x \widetilde{K}_y(r^{*2/3}y)}{K^2}$$

$$\times e^{-\int_0^y dy' 2\nu_2^*[\sqrt{\dot{\gamma}^*}\widetilde{K}(r^{*2/3}y')/r^*][\widetilde{K}(r^{*2/3}y')^2/r^{*4/3}]}. \tag{E34}$$

We should note that

$$\nu_2^*\left(\frac{\sqrt{\dot{\gamma}^*}\widetilde{K}(r^{*2/3}y')}{r^*}\right) \simeq \nu_2^*(0) + O(r^{*-4/3}) \tag{E35}$$

can be used for $r^* \gg 1$ from Eq. (E11). Using this equation and Eq. (E12), we obtain

$$\bar{\Delta}_3(r) = J_3 r^{*-5/3} + O(r^{*-2/3}), \tag{E36}$$

where

$$J_3 = \dot{\gamma}^{*-1} l_c^{-3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K} \int_0^\infty dY \frac{K_x^2 Y}{K^2} e^{-2\nu_2^*(0)K_x^2 Y^3/3}. \tag{E37}$$

Therefore, the asymptotic form of $\bar{\Delta}_3(r)$ is given by Eq. (61).

4. Asymptotic form of $\bar{\Delta}_4(r)$

$\bar{\Delta}_4(r)$ can be rewritten as

$$\widetilde{\Delta}_4(\boldsymbol{k}) = \sigma^{-3} \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\sin(kr/\sigma)}{kr/\sigma}$$

$$\times \int_0^{\bar{t}} d\bar{t} F_4(\boldsymbol{k},\bar{t}) e^{-\int_0^{\bar{t}} d\bar{s} 2\nu_2^*[\bar{k}(\dot{\gamma}^* \bar{s})]\widetilde{k}(\dot{\gamma}^* \bar{s})^2}, \tag{E38}$$

where we have introduced $F_4(\boldsymbol{k},t) = \{\frac{k_x \hat{k}_y(\dot{\gamma}^* \bar{t})}{\hat{k}(\dot{\gamma}^* \bar{t})^2} F(\boldsymbol{k},\bar{t}) - \frac{k_z}{\hat{k}(\dot{\gamma}^* \bar{t})}\} F(\boldsymbol{k},\bar{t})$. From the transformations $\bar{t} = r^{*2/3}Y/\dot{\gamma}^*$, $\bar{s} = r^{*2/3}Y'/\dot{\gamma}^*$, and $\boldsymbol{k} = \sqrt{\dot{\gamma}^*}K/r^*$, we obtain

$$\widetilde{\Delta}_4(\boldsymbol{k}) = \dot{\gamma}^{*-1} l_c^{-3} r^{*-7/3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$

$$\times \int_0^\infty dY F_4(\sqrt{\dot{\gamma}^*}K/r^*, r^{*2/3}Y/\dot{\gamma}^*)$$

$$\times e^{-\int_0^Y dY' 2\nu_2^*[\sqrt{\dot{\gamma}^*}\widetilde{K}(r^{*2/3}Y')/r^*][\widetilde{K}(r^{*2/3}Y')^2/r^{*4/3}]}. \tag{E39}$$

Substituting the asymptotic expression

$$F_4(\sqrt{\dot{\gamma}^*}K/r^*, r^{*2/3}Y/\dot{\gamma}^*) \simeq \left(\frac{\pi K_Z}{2K_\perp} + \frac{|K_x|}{K}M(\boldsymbol{K})\right)^2 r^{*2/3}Y \tag{E40}$$

for $r^* \gg 1$, Eqs. (E11) and (E12) into Eq. (E39), we obtain

$$\bar{\Delta}_4(r) = J_4 r^{*-5/3} + O(r^{*-2/3}), \tag{E41}$$

where

$$J_4 = \dot{\gamma}^{*-1} l_c^{-3} \int \frac{d\boldsymbol{K}}{(2\pi)^3} \frac{\sin K}{K}$$

$$\times \int_0^\infty dY \left(\frac{\pi K_Z}{2K_\perp} + \frac{|K_x|}{K}M(\boldsymbol{K})\right)^2 Y e^{-2\nu_2^*(0)K_x^2 Y^3/3}. \tag{E42}$$

Hence, we find that the asymptotic form of $\bar{\Delta}_4(r)$ is given by Eq. (62).

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